

SOME REMARKS ON THE SUPER FERMAT PROBLEM FOR BINARY FORMS

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To James Joseph (Sylvester),

ABSTRACT. The space $Pol_d \simeq \mathbb{CP}^d$ of all complex-valued binary forms of degree d (considered up to a constant factor) has a standard stratification, each stratum of which contains all forms with the multiplicities of their distinct roots given by a fixed partition $\mu \vdash d$. For each such stratum S_μ , we introduce its secant degeneracy index ℓ_μ which is the minimal number of projectively dependent pairwise distinct points on S_μ , i.e., points whose projective span has dimension smaller than $\ell_\mu - 1$. In what follows, we initiate the study of the secant degeneracy index ℓ_μ and clarify its relation to the super Fermat problem for binary forms.

1. INTRODUCTION

In what follows by a form we will always mean a binary form. The standard stratification of the d -dimensional projective space Pol_d of all complex-valued binary forms of degree d (considered up to a non-vanishing constant factor) according to the multiplicities of their distinct roots is a well-known and widely used construction in mathematics, see e.g. [Ar, Va, KhSh]. Its strata denoted by S_μ are enumerated by partitions $\mu \vdash d$. Cohomology of S_μ with different coefficients appear in different contexts and were intensively studied over the years, see e.g. [Va]. It seems however that Problem 2 below which has an immediate application to the so-called super Fermat problem for binary forms going back to J. J. Sylvester¹, has not been earlier discussed. (When working with binary forms of degree d , we will consider their zero loci as positive divisors of degree d on \mathbb{CP}^1 .)

Definition 1. Given a positive-dimensional quasiprojective variety $V \subset \mathbb{CP}^d$, we define its *secant degeneracy index* ℓ_V as the minimal positive integer ℓ such that there exists ℓ distinct points on V which are projectively dependent, i.e. whose projective span has dimension at most $\ell - 2$.

Remark 1. Obviously, $3 \leq \ell_V \leq d + 2$. The upper bound is attained, for example, for a parabola in the plane or, more generally, for a rational normal curve $V \subset \mathbb{CP}^d$. On the other hand, if V contains $\mathbb{CP}^1 \setminus \{\text{finite set}\}$, then $\ell_V = 3$. To the best of our knowledge, the invariant ℓ_V has not been previously studied. However a related

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¹James Joseph Sylvester was born as James Joseph in a wealthy Jewish family in London in 1814. He took an extra name Sylvester due to US immigration requirements and in order to avoid problems related to his ethnic origin which however has not helped him that much.

question about the uniqueness of representation of generic forms of subgeneric rank as sums of powers of linear forms has been studied in e.g. [COV].

Denote by $\ell_{\bar{V}}$ the secant degeneracy index of the closure $\bar{V} \subset \mathbb{C}P^d$. Obviously, $\ell_{\bar{V}} \leq \ell_V$. The latter inequality can be strict as shown by Example 1 in § 2.

Problem 1. *For which (non-closed) quasi-projective varieties $V \subset \mathbb{C}P^d$, $\ell_V = \ell_{\bar{V}}$?*

The principal question considered in the present paper is as follows.

Problem 2. *For a given partition $\mu \vdash d$, calculate/estimate its secant degeneracy indices $\ell_\mu := \ell_{S_\mu}$ and $\ell_{\bar{\mu}} := \ell_{\bar{S}_\mu}$.*

For a given partition μ , the equation

$$(1) \quad f_1 + f_2 + \cdots + f_{\ell_\mu} = 0,$$

is called the *minimal secant degeneracy relation* for μ . A solution of the latter equation is a collection of pairwise non-proportional forms from S_μ satisfying (1).

We will now explain that Problem 2 is a natural generalisation of the so-called super Fermat problem for binary forms. Recall that a super Fermat equation

$$(2) \quad u_1^k + u_2^k + \cdots + u_l^k = u^k,$$

is an analog of the Fermat equation with l terms in the left-hand side instead of two, where $l > 2$. Solutions of (2) in positive integers have been already studied by L. Euler [Eu] who conjectured that, for any positive integer k , the number $G_0(k)$ defined as the minimal value of l for which (2) has a solution in positive integers, satisfies the inequality $G_0(k) \geq k$.

Euler's conjecture was disproved two centuries later by R. Frye who found the counterexample $95800^4 + 217519^4 + 414560^4 = 422481^4$, for $k = 4$. N. Elkies found that $2682440^4 + 15365634^4 + 18796760^4 = 201615673^4$ which also shows that $G_0(4) = 3$ and disproves Euler's conjecture, see [El]. A little earlier L. J. Lander and T. R. Parkin [LaPa] found the counterexample to Euler's conjecture for $k = 5$, namely $27^5 + 84^5 + 110^5 + 133^5 = 144^5$. Thus, $G_0(5) = 3$ or 4.

For higher values of k , Euler's conjecture is still open. For $k = 6$, an interesting example $74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6 = 1141^6$ is contained in [LaPaSe]; it shows that $3 \leq G_0(6) \leq 7$. There are no corresponding examples for $k > 6$.

An analog of the super Fermat problem for binary forms (and other classes of functions) has also been studied at least since the late 19-th century, see [Li]. The main question under consideration in this area is as follows.

Problem 3. *Given positive integers k, d , find/estimate the minimal number $\ell(d, k)$ such that there exist pairwise non-proportional binary forms $f_1, f_2, \dots, f_{\ell(d, k)}$ of degree d satisfying the super Fermat equation*

$$(3) \quad f_1^k + f_2^k + \cdots + f_{\ell(d, k)}^k = 0.$$

Obviously, $\ell(d, 1) = 3$ and $\ell(1, k) = k + 2$. Substituting $x \mapsto x^d$ and $y \mapsto y^d$, in a solution for $d = 1$, we conclude that $\ell(d, k) \leq k + 2$. A known result of Liouville implies that $\ell(d, k) > 3$ for $k \geq 3$, see [Li] and [Ri].



FIGURE 1. Two main heroes of this story: Professor J. J. Sylvester and Professor B. Reznick. They look rather alike, don't they?

As an illustration, let us briefly discuss the case $d = 2$. Pythagoras' theorem inspires the example

$$(x^2 + y^2)^2 + (i(x^2 - y^2))^2 + (2ixy)^2 = 0,$$

giving $\ell(2, 2) = 3$ which implies $\ell(d, 2) = 3$ for $d \geq 2$. An identity

$$(6x^2 - 4xy + 4y^2)^3 = (3x^2 + 5xy - 5y^2)^3 + (4x^2 - 4xy + 6y^2)^3 + (5x^2 - 5xy - 3y^2)^3,$$

discovered by S. Ramanujan in 1913 implies that $\ell(2, 3) = 4$ which, in its turn, gives $\ell(d, 3) = 4$ for $d \geq 2$. Another example of this kind which can be found in [Re] is as follows

$$(x^2 + xy - y^2)^3 + (x^2 - xy - y^2)^3 = 2(x^2)^3 - 2(-y^2)^3.$$

Next $\ell(2, 4) = \ell(2, 5) = 4$ (implying $\ell(d, 4) = 4$ and $\ell(d, 5) = 4$, for $d \geq 2$) which is proven by

$$(x^2 + y^2)^4 + (\omega x^2 + \omega^2 y^2)^4 + (\omega^2 x^2 + \omega y^2)^4 = 18(xy)^4$$

(where ω is a nontrivial cubic root of 1) and

$$\sum_{j=0}^3 (-1)^j (i^j x^2 + \sqrt{-2}xy + i^{-j}y^2)^5 = 0.$$

Additionally in [Re], B. Reznick shows that $\ell(2, 6) = \ell(2, 7) = 5$ which is proven by

$$\sum_{j=0}^3 \left(i^{-j}x^2 + \sqrt{-\frac{2}{5}}xy + i^k y^2 \right)^6 = -\frac{5632}{125}x^6y^6$$

and

$$\sum_{j=0}^3 \left(i^{-j}x^2 + \sqrt{-\frac{6}{5}}xy + i^k y^2 \right)^7 = -\frac{2^{23/2}3^{1/2}13}{5^{7/2}}ix^7y^7.$$

[Re] contains the equalities showing that $\ell(2, 9) \leq 6$ and $\ell(2, 14) = 6$. The latter fact follows from the miraculous identity $\sum_{j=0}^5 q_j^{14}(x, y) = 0$ where $q_k(x, y) = \zeta_5^k x^2 + ixy + \zeta_5^{-k} y^2$ for $0 \leq k \leq 4$ and $q_5(x, y) = \sqrt{-5}xy$. B. Reznick has also shown that $\ell(2, 2k) \leq k + 2$ and $\ell(2, 2k + 1) \leq k + 3$ (unpublished).

In connection with $\ell(d, k)$, the following question looks very natural.

Problem 4. Determine $L(k) = \lim_{d \rightarrow \infty} \ell(d, k)$.

According to [Ha],

$$(4) \quad \frac{1}{2} + \sqrt{k + \frac{1}{4}} < L(k) \leq \sqrt{4k + 1}.$$

It is quite obvious that Problem 3 is a special case of Problem 2. Namely, $\ell(d, k) = \ell_{\bar{k}^d}$ where k^d is a partition of kd . But Problem 2 has much more flexibility and is solvable for a sufficiently large class of partitions μ .

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2. RESULTS AND CONJECTURES ON THE SECANT DEGENERACY INDEX

2.1. General results on ℓ_μ and $\ell_{\bar{\mu}}$. Recall the notion of the refinement partial order “ \succ ” on the set of all partitions of a given positive integer d . Namely, $\mu' \succ \mu$ in this order if μ' is obtained from μ by merging of some parts of μ . The unique minimal element of this partial order is $(1)^d$, while its unique maximal element is (d) .

Proposition 1. *The number $\ell_{\bar{\mu}}$ is monotone non-decreasing in the refinement partial order. In other words, if $\mu' \succ \mu''$, then $\ell_{\bar{\mu}'} \geq \ell_{\bar{\mu}''}$.*

Proof. Obvious, since $\bar{S}_{\mu'} \subset \bar{S}_{\mu''}$. \square

The following observation is in place here. There are partitions μ for which $\ell_\mu \neq \ell_{\bar{\mu}}$.

Example 1. *For example, for $\mu_d = (2d + 1, 2d, d, d)$, $\ell_{\bar{\mu}_d} = 3$, but ℓ_{μ_d} grows to infinity when $d \rightarrow \infty$.*

However we conjecture the following.

Conjecture 1. *For any partition $\mu \vdash d$, there exists $\mu' \succeq \mu$ such that $\ell_{\bar{\mu}} = \ell_{\mu'}$.*

Another important observation is as follows. Given a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, we call $\nu = (\mu_{i_1} \geq \mu_{i_2} \geq \dots \geq \mu_{i_s})$ where $1 \leq i_1 < i_2 < \dots < i_s \leq r$, a subpartition of μ .

Proposition 2. *For a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$ and any its subpartition ν , $\ell_\mu \leq \ell_\nu$. In particular, $\ell_\mu \leq \mu_r + 2$.*

Proof. Given a subpartition $\nu = (\mu_{i_1} \geq \mu_{i_2} \geq \dots \geq \mu_{i_s})$ of a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, let

$$(5) \quad f_1 + f_2 + \dots + f_{\ell_\nu} = 0,$$

be a linear dependence of pairwise non-proportional binary forms from S_ν realizing its secant degeneracy index. Take the partition $\hat{\mu} = \mu \setminus \nu = (\hat{\mu}_1 \geq \hat{\mu}_2 \geq \dots \geq \hat{\mu}_{r-s})$. Multiplying the latter equality by $\prod_{j=1}^{r-s} (x - a_j y)^{\hat{\mu}_j}$, where a_j are generic complex numbers, we get a linear dependence between polynomials in S_μ . The inequality $\ell_\mu \leq \mu_r + 2$ is a special case of the general inequality, if one chooses $\nu = (\mu_r)$. Observe that for the partition $(d) \vdash d$, $\ell_{(d)} = d + 2$, since the set of binary forms of degree d with a root of multiplicity d is a rational normal curve in $Pol_d \simeq \mathbb{CP}^d$. \square

The latter upper bound $\ell_\mu \leq \mu_r + 2$ is sharp, for example, for any partition with $\mu_r = 1$, but not in general. Namely, already in the case $\mu = (2^2) \vdash 4$, $\ell_\mu = 3 < 4 = \mu_2 + 2$. For $\mu = (3^2) \vdash 6$, $\ell_\mu = 4 < 5$; $\mu = (4^2) \vdash 8$, $\ell_\mu = 4 < 6$.

Before formulating general results about ℓ_μ , let us provide more concrete examples and information about ℓ_μ .

Proposition 3. *Let $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$ be a partition with two different indices i_1 and i_2 such that $\mu_{i_1} - \mu_{i_1+1} = \mu_{i_2} - \mu_{i_2+1} = 1$. Then, $\ell_\mu \leq 4$.*

Proof. Without loss of generality, assume that $i_1 < i_2$, and consider two different cases.

Case 1. $i_2 = i_1 + 1$. Take a subpartition $\nu = (\mu_{i_1}, \mu_{i_1+1}, \mu_{i_1+2}) = (\mu_{i_1+2} + 2, \mu_{i_1+2} + 1, \mu_{i_1+2})$; set $k = \mu_{i_1+2}$. We know that $\ell_\mu \leq \ell_\nu$. So it is enough to prove that $\ell_\nu \leq 4$. Take three distinct complex numbers p, q and r , and consider four polynomials

$$\begin{aligned} g_1 &= (x-p)^{k+2}(x-q)^{k+1}(x-r)^k, & g_2 &= (x-p)^{k+2}(x-r)^{k+1}(x-q)^k, \\ g_3 &= (x-q)^{k+2}(x-p)^{k+1}(x-r)^k, & g_4 &= (x-r)^{k+2}(x-p)^{k+1}(x-q)^k. \end{aligned}$$

A linear combination $ag_1 + bg_2 + cg_3 + dg_4$ is given by

$$Q(x)(a(x-p)(x-q) + b(x-p)(x-r) + c(x-q)^2 + d(x-r)^2),$$

where $Q(x) = (x-p)^{k+1}(x-q)^k(x-r)^k$. Polynomials $(x-p)(x-q)$, $(x-p)(x-r)$, $(x-q)^2$ and $(x-r)^2$ are linearly dependent. Thus there exist a, b, c, d such that $ag_1 + bg_2 + cg_3 + dg_4 = 0$. Hence $\ell_\nu \leq 4$.

Case 2. $i_2 > i_1 + 1$. Take a subpartition $\nu = (\mu_{i_1}, \mu_{i_1+1}, \mu_{i_2}, \mu_{i_2+1}) = (\mu_{i_1+1} + 1, \mu_{i_1+1}, \mu_{i_2+1} + 1, \mu_{i_2+1})$; set $k_1 = \mu_{i_1+1}$ and $k_2 = \mu_{i_2+1}$. We know that $\ell_\mu \leq \ell_\nu$. So it is enough to prove that $\ell_\nu \leq 4$. Take four distinct complex numbers p, q, r and t , and consider four polynomials

$$\begin{aligned} g_1 &= (x-p)^{k_1+1}(x-q)^{k_1-1}(x-r)^{k_2+1}(x-s)^{k_2}, & g_2 &= (x-q)^{k_1+1}(x-p)^{k_1-1}(x-r)^{k_2+1}(x-s)^{k_2}, \\ g_3 &= (x-p)^{k_1+1}(x-q)^{k_1-1}(x-s)^{k_2+1}(x-r)^{k_2}, & g_4 &= (x-q)^{k_1+1}(x-p)^{k_1-1}(x-s)^{k_2+1}(x-r)^{k_2}. \end{aligned}$$

A linear combination $ag_1 + bg_2 + cg_3 + dg_4$ is given by

$$R(x)(a(x-p)(x-r) + b(x-q)(x-r) + c(x-p)(x-s) + d(x-q)(x-s)),$$

where $R(x) = (x-p)^{k_1}(x-q)^{k_1}(x-r)^{k_2}(x-s)^{k_2}$. Polynomials $(x-p)(x-r)$, $(x-q)(x-r)$, $(x-p)(x-s)$ and $(x-q)(x-s)$ are linearly dependent. Thus there exist a, b, c, d such that $ag_1 + bg_2 + cg_3 + dg_4 = 0$, and hence $\ell_\nu \leq 4$. \square

Definition 2. By the *radical* of a given binary form we mean the binary form obtained as the product of all distinct linear factors of the original form.

Proposition 4. *Take any partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$. Given an arbitrary positive integer i , form a partition $\mu' = (\mu_1 + i \geq \mu_2 + i \geq \dots \geq \mu_r + i \geq i, i, \dots, i)$, where the entry i is repeated $r(\ell_\mu - 1)$ times at the end of μ' . Then, $\ell_{\mu'} \leq \ell_\mu$.*

Proof. Let f_1, \dots, f_{ℓ_μ} be a solution of (1). Consider the radical g of polynomial $f_1 f_2 \dots f_{\ell_\mu}$. Degree of g is less than $r\ell_\mu$, because any form f_j has exactly r distinct roots.

Construct g' as the product of g by $r\ell_\mu - \deg(g)$ new distinct linear forms, and set $f'_j = f_j \cdot (g')^i$, for $j = 1, \dots, \ell_\mu$. It is easy to see that every f'_j has the root partition given μ' . Furthermore, we get

$$f'_1 + \dots + f'_{\ell_\mu} = (f_1 + \dots + f_{\ell_\mu}) \cdot (g')^i = 0,$$

hence, $\ell_{\mu'} \leq \ell_\mu$. \square

Corollary 1. *For any partition μ containing the subpartitions: $\nu = (t+1, t, t)$, where t is any positive integer, the secant degeneracy index ℓ_μ equals 3. More generally, for any positive integer t , and any partition μ containing the subpartition: $\nu = (t+i, \underbrace{t, t, \dots, t}_{i+1})$, the secant degeneracy index ℓ_μ is at most $i+2$.*

For a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, we define its jump multiset J_μ as the multiset of all positive numbers in the set $\{\mu_1 - \mu_2, \dots, \mu_{r-1} - \mu_r, \mu_r\}$. We denote by h_μ the minimal (positive) jump of μ , i.e. the element of J_μ with minimal length. Our first general result is as follows.

Theorem 5. *For any $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, $\ell_\mu \geq \sqrt{h_\mu + 1} + 1$.*

Proof. Given μ , let $\{f_1, \dots, f_{\ell_\mu}\}$ be a collection of forms solving (1). Assume that $\{f_1, \dots, f_{\ell_\mu}\}$ gives a counterexample to the statement. Denote by g the GCD of $\{f_1, \dots, f_{\ell_\mu}\}$ and consider the relation

$$\frac{f_1}{g} + \dots + \frac{f_{\ell_\mu}}{g} = 0.$$

For any i , every root of the polynomial $\frac{f_i}{g}$ has multiplicity not smaller than h_μ , where $h_\mu > \ell_\mu$, since otherwise $\{f_1, \dots, f_{\ell_\mu}\}$ is not a counterexample.

Consider the sequence of Wronskians

$$w_i = W\left(\frac{f_1}{g}, \dots, \frac{f_{i-1}}{g}, \frac{f_{i+1}}{g}, \dots, \frac{f_{\ell_\mu}}{g}\right), \quad i = 1, \dots, \ell_\mu.$$

All these Wronskians are proportional to each other due to the latter relation.

Let α be a root of some f_i . There exists s such that $\frac{f_s}{g}$ is not divisible by $(x - \alpha)$, since otherwise g is not the GCD.

For any t , consider the multiplicity of the root of w_t at α . It satisfies the inequality:

$$\text{ord}_\alpha(w_t) \geq \sum \left(\text{ord}_\alpha\left(\frac{f_j}{g}\right) \right) - (\ell_\mu - 2) \# \left\{ i : (x - \alpha) \mid \frac{f_i}{g} \right\},$$

because any column of the Wronski matrix corresponding to $(x - \alpha) \mid \frac{f_i}{g}$ is divisible by $(x - \alpha)^{\text{ord}_\alpha(\frac{f_i}{g}) - t + 2}$.

Hence,

$$\begin{aligned} \deg w_1 &\geq \sum_{i=1}^{\ell_\mu} \left(\deg\left(\frac{f_i}{g}\right) - (\ell_\mu - 2) \#_{\text{roots}}\left(\frac{f_i}{g}\right) \right) = \\ &= \ell_\mu(|\mu| - \deg g) - (\ell_\mu - 2) \sum_{i=1}^{\ell_\mu} \#_{\text{roots}}\left(\frac{f_i}{g}\right). \end{aligned}$$

On the other hand,

$$\deg w_1 \leq (\ell_\mu - 1) \left(\deg\left(\frac{f_i}{g}\right) - \ell_\mu + 2 \right) = (\ell_\mu - 1)(|\mu| - \deg g) - (\ell_\mu - 1)(\ell_\mu - 2).$$

We obtain

$$(\ell_\mu - 1)(|\mu| - \deg g) - (\ell_\mu - 1)(\ell_\mu - 2) \geq \ell_\mu(|\mu| - \deg g) - (\ell_\mu - 2) \sum_{i=1}^{\ell_\mu} \#_{\text{roots}} \left(\frac{f_i}{g} \right),$$

i.e.,

$$(\ell_\mu - 2) \sum_{i=1}^{\ell_\mu} \#_{\text{roots}} \left(\frac{f_i}{g} \right) - (\ell_\mu - 1)(\ell_\mu - 2) \geq |\mu| - \deg g.$$

The number $\#_{\text{roots}} \left(\frac{f_i}{g} \right)$ of distinct roots is at most $\frac{|\mu| - \deg g}{h_\mu}$, because each root has multiplicity at least h_μ . Thus

$$(\ell_\mu - 2)(\ell_\mu - 1) \frac{|\mu| - \deg g}{h_\mu} - (\ell_\mu - 1)(\ell_\mu - 2) \geq |\mu| - \deg g.$$

Hence,

$$(\ell_\mu - 2)(\ell_\mu - 1) > h_\mu.$$

□

2.2. Partitions with growing and stabilising secant degeneracy indices.

Definition 3. Given a partition μ and a positive integer m , a solution of

$$(6) \quad f_1 + f_2 + \cdots + f_m = 0,$$

with pairwise non-proportional $f_i \in S_\mu$ is called a *common radical solution* if all f_i 's have the same radical, i.e. the same set of distinct linear factors (up to a constant factor).

Theorem 6. If $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$ satisfies the inequality

$$m \leq \sqrt{\frac{\mu_r}{r-1}} + 1,$$

then any solution of (6) is a common radical solution.

Proof. Assume the opposite. Let $\{f_1, \dots, f_m\}$ be a solution of (1) which is not a common radical solution. Let g be the GCD of $\{f_1, \dots, f_m\}$.

For the term $f_i = c_i(x - a_{i,1})^{\mu_1} \cdots (x - a_{i,r})^{\mu_r}$, define

$$g_i := (x - a_{i,1})^{\mu_1 - m + 2} \cdots (x - a_{i,r})^{\mu_r - m + 2}.$$

Observe that g_i is a polynomial, because any root of f_i has multiplicity at least $\mu_r > m$.

Consider the sequence of Wronskians

$$w_i = W(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_m), \quad i = 1, \dots, m.$$

They are proportional to each other, because $f_1 + \dots + f_m = 0$. Notice that, for $i \neq t$, the column in the Wronski matrix for w_t corresponding to f_i is divisible by g_i . Hence w_t is divisible by $\frac{\prod_{i=1}^m g_i}{g_t}$.

Since $\{f_1, \dots, f_m\}$ is not a common radical solution, there exists $\alpha \in \mathbb{C}$, such that α is a root of f_p but it is not a root of f_q for some $p \neq q$.

Since the Wronskians w_p and w_q are proportional, they are divisible by

$$\text{LCM} \left(\frac{\prod_{i=1}^m g_i}{g_p}, \frac{\prod_{i=1}^m g_i}{g_q} \right) = \frac{\prod_{i=1}^m g_i}{\text{GCD}(g_p, g_q)} = \frac{\prod_{i=1}^m g_i}{g_p} \frac{g_p}{\text{GCD}(g_p, g_q)}.$$

Then these Wronskians are divisible by $\frac{\prod_{i=1}^m g_i}{g_p}(x - \alpha)^{\mu_r - m + 2}$. Therefore their degrees are greater than or equal to

$$(m-1)(|\mu| - r(m-2)) + \mu_r - m + 2.$$

On the other hand, the degrees of the Wronskians are at most $(m-1)(|\mu| - m + 2)$. Thus,

$$(m-1)(|\mu| - m + 2) \geq (m-1)(|\mu| - r(m-2)) + \mu_r - m + 2,$$

which implies $-(m-1)(m-2) \geq -r(m-1)(m-2) + \mu_r - m + 2$. After straightforward simplifications the latter inequality gives

$$m-1 \geq \sqrt{\frac{\mu_r}{r-1}}.$$

Contradiction. \square

Corollary 2. For $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, either $\ell_\mu \geq \sqrt{\frac{\mu_r}{r-1}} + 1$ or any solution of (1) is a common radical solution.

Notation. For a given partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, define

$$\mu^{(t)} := (\mu_1 + t \geq \mu_2 + t \geq \dots \geq \mu_r + t).$$

Definition 4. We say that a partition μ has a *growing secant degeneracy index* if $\lim_{t \rightarrow \infty} \ell_{\mu^{(t)}} = +\infty$ and that μ has a *stabilising secant degeneracy index* otherwise.

The following proposition is clear.

Proposition 7. A partition $\mu = (\mu_1 \leq \mu_2 \leq \dots \leq \mu_r) = (i_1^{m_1}, i_2^{m_2}, \dots, i_s^{m_s})$ with $i_1 > i_2 > \dots > i_s$, has a stabilising secant degeneracy index if and only if, for some positive integer m , there exists a common radical solution of (6). A partition μ has a growing secant degeneracy index if and only if the linear span of the Sym_r -orbit of any form $f \in S_\mu$ has the dimension equal to the multinomial coefficient $\frac{r!}{m_1!m_2!\dots m_s!}$. (Here the symmetric group Sym_r acts on any $f \in S_\mu$ by permuting all its r distinct roots.)

Remark 2. For any partition μ with a growing secant degeneracy index, i.e., for $\ell_{\mu^{(t)}} \rightarrow \infty$, we know that

$$\sqrt{\frac{\mu_r + t}{r-1}} + 1 \leq \ell_{\mu^{(t)}} \leq \mu_r + t + 2,$$

see Proposition 2 and Theorem 6.

Conditions formulated in Proposition 7 are difficult to check which motivates the following questions.

Problem 5. Give necessary and sufficient combinatorial conditions for a partition μ to belong to one of the above two classes.

Problem 6. For any partition μ with a growing secant degeneracy index, what is the leading term of the asymptotic of $\ell_{\mu^{(t)}}$, when $t \rightarrow +\infty$? Does it depend on a particular choice of μ ?

2.3. More on partitions with stabilising secant degeneracy index. Here we give a simple sufficient condition for μ to have a stabilising secant degeneracy index.

Proposition 8. *Take a multiset $\tau_\mu = \{\mu_1, \dots, \mu_r\}$ and a sequence $\{a_1, \dots, a_r\}$ with the following properties:*

- (i) for any i , $a_i \in [1, r]$;
- (ii) the number of different permutation π of τ_μ such that $\pi_i \geq a_i$ is at least $|\mu| - \sum_{i=1}^r a_i + 2$.

Then there exists a solution of (6) with a common radical. Furthermore there is such a solution with an arbitrary set of r distinct roots.

Proof. Let $f = (x - c_1y)^{\mu_1} \cdot (x - c_2y)^{\mu_2} \cdot \dots \cdot (x - c_ry)^{\mu_r}$ be any function from S_μ . Consider the set $\mathcal{D}_{(a_1, \dots, a_r)}$ of permutations of multiset τ_μ , which satisfy the parking-type conditions of Theorem 8. For any $\pi \in \mathcal{D}_{(a_1, \dots, a_r)}$, define f_π as a function of Sym_r -orbit of f corresponding to π . Any such function f_π is divisible by $g = (x - c_1y)^{a_1} \cdot (x - c_2y)^{a_2} \cdot \dots \cdot (x - c_ry)^{a_r}$, because π satisfies the above parking-type conditions.

For any $\pi \in \mathcal{D}_{(a_1, \dots, a_r)}$, define $\hat{f}_\pi := \frac{f_\pi}{g} \in S_{|\mu| - \sum_{i=1}^r a_i}$. If $|\mathcal{D}_{(a_1, \dots, a_r)}| \geq |\mu| - \sum_{i=1}^r a_i + 2$, then the forms \hat{f}_π are linearly dependent. Therefore, the forms f_π , for $\pi \in \mathcal{D}_{(a_1, \dots, a_r)}$, are also linearly dependent. \square

2.4. More on partitions with growing secant degeneracy index. Here we give a sufficient condition for μ to have a growing secant degeneracy index.

Corollary 3. *Any partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, such that every its jump is at least $(r!)^2$, has a growing secant degeneracy index.*

Proof. Assume that $\ell_{\mu^{(t)}}$ does not grow to infinity. Then by Theorem 5,

$$\ell_{\mu^{(t)}} \geq \sqrt{h_\mu^t + 1} + 1 \geq \sqrt{h_\mu^0 + 1} + 1 \geq \sqrt{(r!)^2 + 1} + 1 > r!.$$

However the number of different polynomials (up to a constant factor) with fixed r roots and their multiplicities μ^t is at most $r!$. Hence no common radical solution might exist. Contradiction. \square

The next proposition shows that if there are many jumps of small size, then the secant degeneracy index is bounded.

Proposition 9. *Let d be a positive integer exceeding 45. If for a partition $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$, the number of jumps of sizes less than or equal to d is at least $2(\log_2(d) + \log_2(\log_2(d)) + 2)$, then $\ell_\mu \leq d(\log_2(d) + \log_2(\log_2(d))) + 2$.*

Proof. Assume that there are at least $2(\log_2(d) + \log_2(\log_2(d)) + 2)$ such jumps. Consider every second such jump; their number is at least $t = \lceil \log_2(d) + \log_2(\log_2(d)) + 2 \rceil$. Assume that they occupy the positions $j_1 < \dots < j_t$, i.e. $(\mu_{j_i} - \mu_{j_{i+1}}) \leq d$, for $i \in [1, t]$. Furthermore $j_i + 1 < j_{i+1}$, because there is a nontrivial jump between them.

Consider the set of parking permutations of $\tau_\mu = \{\mu_1, \dots, \mu_r\}$ such that:

- $i \notin \{j_1, \dots, j_t\}$, $\pi_i \geq \mu_i$;
- $i \in \{j_1, \dots, j_t\}$, $\pi_i \geq \mu_{i+1}$.

The number of such permutations is 2^t . By Proposition 8, there are solution of size $\sum_{i=1}^t (\mu_{j_i} - \mu_{j_i+1}) + 2 \leq d \cdot t + 2$, because

$$d \cdot t + 2 \leq d \cdot (\log_2(d) + \log_2(\log_2(d)) + 2) + 2 \leq d \cdot (\log_2(d) + \log_2(\log_2(d)) + 3) \leq d \cdot 2 \cdot \log_2(d) \leq 2^{\log_2(d) + \log_2(\log_2(d)) + 1} \leq 2^t$$

which finishes the proof of the proposition. \square

2.5. Examples. It is obvious that all partitions with two parts have growing secant degeneracy index.

Proposition 10. (i) For partitions $\mu = (p, 2)$, one has that $\ell_\mu = 3$ when $p = 2$, and $\ell_\mu = 4$ when $p > 2$.

(ii) For partitions $\mu = (p, 3)$, one has that $\ell_\mu = 4$ when $p = 3, 4, 5$ and $\ell_\mu = 5$ when $p > 7$. Cases $\mu = (6, 3)$ and $\mu = (7, 3)$ are still open.

Proof. In case $\mu = (3, 4)$ we found the example:

$$y^3(x+y)^4 - y^3x^4 = L(x+ay)^3y^4 + (1-L)(x+by)^3y^4,$$

where $a = 3 - \sqrt{3}$, $b = 3 + \sqrt{3}$ and $L = \frac{9-5\sqrt{3}}{18}$;

In case $\mu = (5, 3)$ we found the example:

$$f_1 + f_2 - f_3 - f_4 = 0,$$

where $f_1(x) = (x + c_1^5y)^3(x + c_1^{-3}y)^5$; $f_2(x) = (x + c_2^5y)^3(x + c_2^{-3}y)^5$; $f_3(x) = (x + c_1^{-5}y)^3(x + c_1^3y)^5$; $f_4(x) = (x + c_2^{-5}y)^3(x + c_2^3y)^5$. Here

$$c_1 = -c_2 = \left(\frac{1 + i\sqrt{35}}{6} \right)^{\frac{1}{4}}.$$

\square

We say that a partition has a *strongly stabilising secant degeneracy index* if a stabilising solution exists for any choice of distinct roots. Using computer algebra packages we were able to prove the following statement.

Lemma 11. *For a partition μ with three parts $a \leq b \leq c$ the following three conditions are equivalent:*

- (i) μ has a stabilising secant degeneracy index;
- (ii) μ has a strongly stabilising secant degeneracy index;
- (iii) the triple $a \leq b \leq c$ belongs to one of the following three types: $a = b$, $c = a + 1$; $b = a + 1$, $c = a + 2$; $b = a + 1$, $c = a + 3$.

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